UNIT-2

(Lecture-6)

Design of Infinite Impulse Response Digital Filters: Design of Digital Butterworth Filter

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Design of Digital Butterworth Filter

Example: Determine H(z) for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq \pi/2$$
$$|H(e^{j\omega})| \leq 0.2 \qquad 3\pi/4 \leq \omega \leq \pi$$

With T = 1s. Apply impulse invariant transformation.

Solution:

Given $\delta_1 = \sqrt{0.5} = 0.707$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$

Using impulse invariant transformation

$$\Omega_1 = \frac{\omega_1}{T} = \frac{\pi}{2} \quad \text{and} \quad \Omega_2 = \frac{\omega_2}{T} = \frac{3\pi}{4}$$

Therefore $\Omega_2 / \Omega_1 = 1.5$

Design of Digital Butterworth Filter

From equation (6), determine order of the filter

$$N \geq \frac{1}{2} \frac{\log \left\{[(1/\delta_2^2 - 1)/(1/\delta_1^2 - 1)\right\}}{\log (\Omega_2/\Omega_1)}$$

$$= \frac{1}{2} \frac{\log \{24/1\}}{\log (1.5)} = 3.91$$

Therefore N = 4

From eq. (7), determine 3dB cut-off frequency

$$\Omega_c = \frac{\Omega_1}{\left[(1/\delta_1^2) - 1\right]^{1/2N}} = \frac{\pi/2}{\left[(1/0.707^2) - 1\right]^{1/8}} = \frac{\pi}{2}$$

From eq. (8), determine $H_a(s)$

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k \, \Omega_c^2}{s^2 + b_k \, \Omega_c \, s + c_k \, \Omega_c^2}$$

$$= \left(\frac{B_1 \, \Omega_c^2}{s^2 + b_1 \, \Omega_c \, s + c_1 \, \Omega_c^2}\right)\left(\frac{B_2 \, \Omega_c^2}{s^2 + b_2 \, \Omega_c \, s + c_2 \, \Omega_c^2}\right)$$

Design of Digital Butterworth Filter

From eq. (10)

$$b_1 = 2\sin\frac{\pi}{8} = 0.76536, \quad c_1 = 1$$

$$b_2 = 2\sin\frac{3\pi}{8} = 1.84776, \quad c_2 = 1$$

$B_1 B_2 = 1$. Therefore $B_1 = B_2 = 1$.

Therefore,

$$H(s) = \left(\frac{2.467}{s^2 + 1.2022\,s + 2.467}\right)\left(\frac{2.467}{s^2 + 2.9025 + 2.467}\right)$$

Using partial fractions,

$$H(s) = \left(\frac{A\,s + B}{s^2 + 1.2022\,s + 2.467}\right) + \left(\frac{C\,s + D}{s^2 + 2.9025 + 2.467}\right)$$

Design of Digital Butterworth Filter

Compare the two expressions, we get

$$6.086 = (s^2 + 2.9025s + 2.467)(As + B) + (s^2 + 1.2022s + 2.467)(Cs + D)$$

Comparing the coefficients of s^3, s^2, s and the constants, we get a set of simultaneous equations.

$$A + C = 0$$
$$2.9025\,A + B + 1.2022\,C + D = 0$$
$$2.467\,A + 2.9025\,B + 2.467\,C + 1.2022\,D = 0$$
$$B + D = 2.467$$

Solving, we get $A = -1.4509$, $B = -1.7443$, $C = 1.4509$ and $D = 4.2113$.

Design of Digital Butterworth Filter

Therefore

$$H(s) = -\left(\frac{1.4509\,s + 1.7443}{s^2 + 1.2022\,s + 2.467}\right) + \left(\frac{1.4509\,s + 4.2113}{s^2 + 2.9025\,s + 2.467}\right)$$

Let $H(s) = H_1(s) + H_2(s)$, where

$$H_1(s) = -\left(\frac{1.4509\,s + 1.7443}{s^2 + 1.2022s + 2.467}\right) \text{ and } H_2(s) = \left(\frac{1.4509\,s + 4.2113}{s^2 + 2.9025\,s + 2.467}\right)$$

Rearranging $H_1(s)$ into the standard form,

$$H_1(s) = -\left(\frac{1.4509\,s + 1.7443}{s^2 + 1.2022\,s + 2.467}\right)$$

$$= -1.4509\left(\frac{s + 1.2022}{(s + 0.601)^2 + 1.451^2}\right)$$

$$= (-1.4509)\left[\frac{s + 0.601}{(s + 0.601)^2 + 1.451^2} + \frac{0.601}{(s + 0.601)^2 + 1.451^2}\right]$$

$$= (-1.4509)\left(\frac{s + 0.601}{(s+0.601)^2 + 1.451^2}\right) - (0.601)\left(\frac{1.451}{(s+0.601)^2 + 1.451^2}\right)$$

Design of Digital Butterworth Filter

Similarly, $H_2(s)$ can be written as

$$H_2(s) = (1.4509)\left(\frac{s+1.45}{(s+1.45)^2+0.604^2}\right) + (3.4903)\left(\frac{0.604}{(s+1.45)^2+0.604^2}\right)$$

Step (v) Determination of $H(z)$. Using Eqs. 8.27 and 8.28,

$$H_1(z) = (-1.4509)\frac{1-e^{-0.601T}(\cos 1.451T)\,z^{-1}}{1-2e^{-0.601T}(\cos 1.451T)\,z^{-1}+e^{-1.202T}\,z^{-2}}$$

$$-(0.601)\frac{e^{-0.601T}(\sin 1.451T)\,z^{-1}}{1-2e^{-0.601T}(\cos 1.451T)\,z^{-1}+e^{-1.202T}\,z^{-2}}$$

and

$$H_2(z) = (1.4509)\frac{1-e^{-1.45T}(\cos 0.604T)\,z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)\,z^{-1}+e^{-2.9T}\,z^{-2}}$$

$$+(3.4903)\frac{e^{-1.45T}(\sin 0.604T)\,z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)\,z^{-1}+e^{-2.9T}\,z^{-2}}$$

where $H(z) = H_1(z) + H_2(z)$.

Upon simplifying we get,

$$H(z) = \frac{-1.4509-0.2321z^{-1}}{1-0.1310z^{-1}+0.3006z^{-2}} + \frac{1.4509+0.1848z^{-1}}{1-0.3862z^{-1}+0.055z^{-2}}$$

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